1. A Position operator $\hat{X}$ is a Hermitian operator, so its eigenvectors form a basis of its corresponding space. But what are their eigenvectors? No one is able to find them within the regular Hilbert space, i.e.

$$\boxed{\hat{X}|?\rangle = x|?\rangle}$$

2. Any function $\phi(x)$ can be expressed by its basis $u_n(x)$ as

$$\phi(x) = \sum c_n u_n(x).$$

On the other hand, from the properties of Dirac's δ-function

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0), \qquad \int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$

and the Fourier's transformation, we may have

$$\phi(x) = \int_{-\infty}^{\infty} dx'\phi(x')\delta(x-x'), \qquad \phi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} dp\phi(p)e^{ipx/\hbar}$$

where $\phi(x')$ and $\phi(p)$ can be treated as the expansion coefficients c_n. Then, one may build up the following relations

Basis (irregular function)	$\leftrightarrow$	State space (non Hilbert space)
$\delta(x-x')$	$\leftrightarrow$	$\|x\rangle$
$(1/\sqrt{2\pi\hbar})e^{ipx/\hbar}$	$\leftrightarrow$	$\|p\rangle$

3. The collection of $|x\rangle$ forms a basis of its corresponding space, named as the **position space**. By the similarity to the finite dimensional vector space, one has

$$\hat{1} = \sum_{i=1}^{n} |e_i\rangle\langle e_i|, \qquad \boxed{\hat{1} = \int_{-\infty}^{\infty} dx|x\rangle\langle x|}$$

$$\Rightarrow \qquad \boxed{|\phi\rangle = \int_{-\infty}^{\infty} dx|x\rangle\langle x|\phi\rangle}$$

where we should not treat $\langle x|\phi\rangle$ as an inner product, because $|x\rangle$ and $|\phi\rangle$ do not belong to the same space. It is a kind of expansion coefficient. Similarly, we may also have

$$\langle\phi| = \int_{-\infty}^{\infty} dx'\langle\phi|x'\rangle\langle x'|$$

4. By the requirement of having the finite inner product of any state $|\phi\rangle$ to itself, say 1, then, with the definition of the inner product of a regular function, we obtain

$$\langle\phi|\phi\rangle \equiv \int_{-\infty}^{\infty} \phi^*(x)\phi(x)dx = 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdx'\langle\phi|x'\rangle\langle x'|x\rangle\langle x|\phi\rangle$$

The only way to make the above equation meaningful is to set

$$\boxed{\langle x'|x\rangle = \delta(x-x').}$$

so that

$$\int_{-\infty}^{\infty} \phi^*(x)\phi(x)dx = 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdx'\langle\phi|x'\rangle\delta(x-x')\langle x|\phi\rangle$$
$$= \int_{-\infty}^{\infty} dx\langle x|\phi\rangle(\int_{-\infty}^{\infty} dx'\langle\phi|x'\rangle\delta(x-x')) = \int_{-\infty}^{\infty} dx\langle\phi|x\rangle\langle x|\phi\rangle$$

Furthermore, from two ends of the above equality, we may also reach the meaning of the expansion coefficient or the "inner product", in non-strict sense, $\langle x|\phi\rangle$, which is

$$\boxed{\langle x|\phi\rangle \equiv \phi(x)}$$

5. The physical application of

$$\boxed{\hat{X}|x\rangle = x|x\rangle}, \qquad \text{or} \qquad \hat{X}|x'\rangle = x'|x'\rangle$$

can be seen as follows

$$\boxed{\hat{X}\phi(x)} = \hat{X}\langle x|\phi\rangle = \hat{X}^\dagger\langle x|\phi\rangle = \langle x|\hat{X}^\dagger|\phi\rangle = \boxed{\langle x|\hat{X}|\phi\rangle} = \int_{-\infty}^{\infty}\langle x|\hat{X}|x'\rangle\langle x'|\phi\rangle dx'$$
$$= \int_{-\infty}^{\infty} x'\langle x|x'\rangle\phi(x')dx' = \int_{-\infty}^{\infty} x'\phi(x')\delta(x-x')dx' = \boxed{x\phi(x)}.$$

6. The representation of a momentum operator in the position space:

$$\boxed{\hat{P}\phi(x)} = \hat{P}\langle x|\phi\rangle = \hat{P}^\dagger\langle x|\phi\rangle = \langle x|\hat{P}^\dagger|\phi\rangle = \boxed{\langle x|\hat{P}|\phi\rangle} = \int_{-\infty}^{\infty}\langle x|\hat{P}|p\rangle\langle p|\phi\rangle dp$$

$$= \int_{-\infty}^{\infty} p\langle x|p\rangle\phi(p)dp = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} p\phi(p)e^{ipx/\hbar}dp = \frac{\hbar}{i}\frac{d}{dx}(\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\phi(p)e^{ipx/\hbar}dp) = \boxed{-i\hbar\frac{d}{dx}\phi(x)}$$

$\Rightarrow \quad \boxed{\hat{P} = -i\hbar\frac{d}{dx}}$, where the plane wave amplitude $\boxed{\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}}$ is applied.

National Taiwan Normal University

Quantum Mechanics - Quiz 1 *11:50-12:10* *11-1-2016*

1. In a 3-dimensional real space R^3, find out the matrix representation of the inversion with respect to the origin in R^3 space. (35%)

2. In R^2 space, the original and new base are as follows, respectively.

$$b = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}, \quad d = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

(a) What is the transition matrix from b to d? (25%)

(b) If a state

$$|\alpha\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

in basis b, what is the form of $|\alpha\rangle$ in basis d? (40%)

National Taiwan Normal University I.D. #:

Quantum Mechanics - Quiz 1 *11:50-12:10* *11-1-2016* Name:

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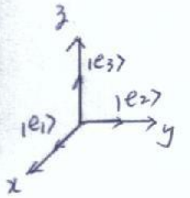

$$|e_1\rangle \rightarrow -|e_1\rangle = -|e_1\rangle + 0|e_2\rangle + 0|e_3\rangle$$
$$|e_2\rangle \rightarrow -|e_2\rangle = 0|e_1\rangle - |e_2\rangle + 0|e_3\rangle$$
$$|e_3\rangle \rightarrow -|e_3\rangle = 0|e_1\rangle + 0|e_2\rangle - |e_3\rangle$$

$$\vec{T} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(若直接寫答案，無|e₁>,|e₂>,|e₃>在反射作用後之結果，只給15分)

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(a) $$|d_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = P_{11}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + P_{21}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} P_{11} \\ -P_{21} \end{pmatrix} \Rightarrow P_{21} = -1,\ P_{11} = 0$$

$$|d_2\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = P_{12}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + P_{22}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} P_{12} \\ -P_{22} \end{pmatrix} \Rightarrow P_{22} = -1,\ P_{12} = 1$$

Transition matrix $= P = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$

(b) ① $$|\alpha\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix}_b = 1\cdot\underset{|b_1\rangle}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + 2\underset{|b_2\rangle}{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = a\underset{|d_1\rangle}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} + c\underset{|d_2\rangle}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} c \\ a+c \end{pmatrix} \Rightarrow \begin{matrix} c = 1 \\ a = -3 \end{matrix}$$

故 $|\alpha\rangle = \begin{pmatrix} -3 \\ 1 \end{pmatrix}_d$

或 ② $$|\alpha\rangle_d = P^{-1}_{b\to d}|\alpha\rangle_b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix}_b = \left(\frac{1}{1}\right)\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}_d$$