6.3 角動量量子化在軌道角動量上的應用

因軌道角動量是角動量中的一種特例，我們將應用上節角動量量子化的結果，來證明軌道角動量 $\hat{\boldsymbol{L}}^2$ 之固有值 $\ell(\ell+1)\hbar^2$ 中之 ℓ 必為整數。其**固有狀態 $|j,m\rangle$ ，在位置空間上角度部分的振幅或投影值，將可以球面調和函數 $Y_\ell^m(\theta,\varphi)$ 來表示**，即

$$\langle\theta,\boldsymbol{\varphi}\mid j,m\rangle=Y_\ell^m(\theta,\boldsymbol{\varphi}) \tag{6.41}$$

其中 ℓ 、m 均為整數，且 $-\ell\le m\le\ell$。

在位置空間 $|\vec{\boldsymbol{r}}\rangle=|x,y,z\rangle=|r,\theta,\boldsymbol{\varphi}\rangle$ 中，位置與動量之運算子 $\hat{\boldsymbol{R}}$ 、 $\hat{\boldsymbol{P}}$ 分別對應於

$$\hat{\boldsymbol{R}}\to R \quad , \quad \hat{\boldsymbol{P}}\to -i\hbar\nabla \tag{6.42}$$

故軌道角動量運算子 $\hat{\boldsymbol{L}}\to\hat{\boldsymbol{R}}\times\hat{\boldsymbol{P}}$ 中各分量，由式(6.3)

$$\hat{L}_x=-i\hbar\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right),\ \hat{L}_y=-i\hbar\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right),\ \hat{L}_z=-i\hbar\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right) \tag{6.43}$$

茲以球面坐標 $|r,\theta,\boldsymbol{\varphi}\rangle$ ，而非直角坐標 $|x,y,z\rangle$ ，來表示軌道角動量 $\hat{L}_x$ 、$\hat{L}_y$ 、$\hat{L}_z$ ，因

$$x=r\sin\theta\cos\boldsymbol{\varphi} \quad , \quad y=r\sin\theta\sin\boldsymbol{\varphi} \quad , \quad z=r\cos\theta \tag{6.44}$$

其中 $r>0$，$0\le\theta\le\pi$，$0\le\boldsymbol{\varphi}\le 2\pi$ ；這是由球面坐標 $r,\theta,\boldsymbol{\varphi}$ 表示直角坐標 x,y,z 的關係式。又微小體積單元

$$d^3r=r^2\sin\theta\,dr\,d\theta\,d\boldsymbol{\varphi}=r^2dr\,d\Omega \tag{6.45}$$

其中固體角 $d\Omega=\sin\theta\,d\theta\,d\boldsymbol{\varphi}$。若以 x,y,z 表 $r,\theta,\boldsymbol{\varphi}$ ，則有

$$r^2=x^2+y^2+z^2 \quad , \quad \cos\theta=\frac{z}{r} \quad , \quad \tan\boldsymbol{\varphi}=\frac{y}{x} \tag{6.46}$$

故

$$\frac{\partial r}{\partial x}=\frac{1}{2}(x^2+y^2+z^2)^{-1/2}\cdot 2x=\frac{x}{r} \tag{6.47}$$

同理 $$\frac{\partial r}{\partial y}=\frac{y}{r},\qquad \frac{\partial r}{\partial z}=\frac{z}{r} \tag{6.48}$$

又 $$-\sin\theta\frac{\partial\theta}{\partial x}=\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\cos\theta=\frac{\partial}{\partial x}\cos\theta=z\frac{\partial}{\partial x}(\frac{1}{r})$$

$$=-\frac{z}{r^2}\frac{\partial r}{\partial x}=-\frac{zx}{r^3}=-\frac{1}{r}\cdot\frac{z}{r}\cdot\frac{x}{r}=-\frac{1}{r}\cos\theta\sin\theta\cos\boldsymbol{\varphi} \tag{6.48}$$

所以 $$\frac{\partial\theta}{\partial x}=\frac{\cos\theta\cos\boldsymbol{\varphi}}{r} \tag{6.50}$$

同理 $$-\sin\theta\frac{\partial\theta}{\partial y}=\frac{1}{r}\cos\theta\sin\theta\sin\varphi,\qquad -\sin\theta\frac{\partial\theta}{\partial z}=\frac{1}{r}(1-\cos^2\theta) \tag{6.51}$$

故 $$\frac{\partial\theta}{\partial y}=\frac{\cos\theta\sin\varphi}{r},\qquad \frac{\partial\theta}{\partial z}=-\frac{\sin\theta}{r} \tag{6.52}$$

而 $$\sec^2\boldsymbol{\varphi}\frac{\partial\varphi}{\partial x}=\frac{\partial}{\partial x}\tan\varphi=y\frac{\partial}{\partial x}(\frac{1}{x})=-\frac{y}{x^2} \tag{6.53}$$

因此 $$\frac{\partial\varphi}{\partial x}=-\frac{y}{x^2}\cos^2\boldsymbol{\varphi} \tag{6.54}$$

同理 $$\frac{\partial\boldsymbol{\varphi}}{\partial y}=-\frac{\cos^2\boldsymbol{\varphi}}{x},\qquad \frac{\partial\boldsymbol{\varphi}}{\partial z}=0 \tag{6.55}$$

利用以上的結果，可將對直角坐標 x,y,z 之偏導數，全部轉成對球面坐標 $r,\theta,\boldsymbol{\varphi}$ 之偏導數：

$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}+\frac{\partial\boldsymbol{\varphi}}{\partial x}\frac{\partial}{\partial\boldsymbol{\varphi}}=\frac{x}{r}\frac{\partial}{\partial r}+\frac{\cos\theta\cos\boldsymbol{\varphi}}{r}\frac{\partial}{\partial\theta}-\frac{y}{x^2}\cos^2\boldsymbol{\varphi}\frac{\partial}{\partial\boldsymbol{\varphi}} \tag{6.56}$$

$$\frac{\partial}{\partial y}=\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial y}\frac{\partial}{\partial\varphi}=\frac{y}{r}\frac{\partial}{\partial r}+\frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial\theta}+\frac{\cos^2\boldsymbol{\varphi}}{x}\frac{\partial}{\partial\varphi} \tag{6.57}$$

$$\frac{\partial}{\partial z}=\frac{\partial r}{\partial z}\frac{\partial}{\partial r}+\frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta}+\frac{\partial\varphi}{\partial z}\frac{\partial}{\partial\varphi}=\frac{z}{r}\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{6.58}$$

再利用式(6.43)，遂可用 $r,\theta,\boldsymbol{\varphi}$ 表示出角動量在直角坐標分量之運算子形式

$$
\begin{aligned}
\hat{L}_x &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) \\
&= -i\hbar\left(\cancel{\frac{yz}{r}\frac{\partial}{\partial r}} - \frac{y\sin\theta}{r}\frac{\partial}{\partial \theta} - \cancel{\frac{yz}{r}\frac{\partial}{\partial r}} - \frac{z\cos\theta\sin\varphi}{r}\frac{\partial}{\partial \theta} - \frac{z\cos^2\boldsymbol{\varphi}}{x}\frac{\partial}{\partial \varphi} \right) \\
&= i\hbar\left(\sin\varphi\frac{\partial}{\partial \theta} + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial \varphi} \right)
\end{aligned}
\tag{6.59}
$$

$$
\begin{aligned}
\hat{L}_y &= -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right) \\
&= -i\hbar\left(\cancel{\frac{xz}{r}\frac{\partial}{\partial r}} - \frac{z\cos\theta\cos\varphi}{r}\frac{\partial}{\partial \theta} - \frac{yz}{x^2}\cos^2\boldsymbol{\varphi}\frac{\partial}{\partial \varphi} - \cancel{\frac{xz}{r}\frac{\partial}{\partial r}} + \frac{x\sin\theta}{r}\frac{\partial}{\partial \theta} \right) \\
&= i\hbar\left(-\cos s\varphi\frac{\partial}{\partial \theta} + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial \varphi} \right)
\end{aligned}
\tag{6.60}
$$

$$
\begin{aligned}
\hat{L}_z &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \\
&= -i\hbar\left(\cancel{\frac{xy}{r}\frac{\partial}{\partial r}} + \cancel{\frac{x\cos\theta\sin\varphi}{r}\frac{\partial}{\partial \theta}} \right. \\
&\qquad \left. + \frac{x\cos^2\boldsymbol{\varphi}}{x}\frac{\partial}{\partial \varphi} - \cancel{\frac{xy}{r}\frac{\partial}{\partial r}} - \cancel{\frac{y\cos\theta\cos\varphi}{r}\frac{\partial}{\partial \theta}} + \frac{y^2}{x^2}\cos^2\boldsymbol{\varphi}\frac{\partial}{\partial \varphi} \right) \\
&= -i\hbar\frac{\partial}{\partial \varphi}
\end{aligned}
\tag{6.61}
$$

因此得到

$$
\hat{\boldsymbol{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\boldsymbol{\varphi}^2} \right) \tag{6.62}
$$

$$
\hat{L}_+ = \hat{L}_x + i\hat{L}_y = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\boldsymbol{\varphi}} \right) \tag{6.63}
$$

$$
\hat{L}_- = \hat{L}_x - i\hat{L}_y = \hbar e^{-i\varphi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\boldsymbol{\varphi}} \right) \tag{6.64}
$$

已知 $\hat{\boldsymbol{L}}$ 滿足角動量的定義(6.4)~(6.5)，故由角動量量子化定理 6.1，所作用的狀態空間裡，**$\hat{L}^2$ 和 $\hat{L}_z$ 享有共同之固有狀態 $|\ell, m\rangle$，並可給出對應之固有值 $\ell(\ell+1)\hbar^2$ 與 $m\hbar$。**